Lecture 6:

Recall: Discrete Fourier Transform:

Definition: The 1D discrete Fourier Transform (DFT) of a function $f(k)$, defined at discrete points $k = 0, 1, 2, \ldots, N-1$ is defined as:

$$\hat{f}(m) = \frac{1}{N} \sum_{k=0}^{N-1} f(k)\, e^{-j\frac{2\pi m k}{N}}$$

(where $j = \sqrt{-1}$, $e^{j\theta} = \cos\theta + j\sin\theta$)

The 2D DFT of a $M \times N$ image $g = (g(k,l))_{k,l}$, where $0 \le k \le M-1$, $0 \le l \le N-1$ is defined as:

$$\hat{g}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km}{M} + \frac{ln}{N}\right)}$$

Remark: The inverse of DFT is given by:

$$g(p,q) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{g}(m,n)\, e^{j2\pi\left(\frac{pm}{M} + \frac{qn}{N}\right)}$$

DFT of g

(no $\frac{1}{Mn}$!)

(no -ve sign)

Why is DFT useful in imaging:

1. DFT of convolution:

Recall: $g * w(n,m) = \sum_{n'=0}^{N-1} \sum_{m'=0}^{N-1} g(n-n', m-m')\, w(n', m')$

$(g, m \in M_{N \times M}(\mathbb{R}))$

Then, the DFT $(g * w)(p,q) = MN\, DFT(g)(p,q)\, DFT(w)(p,q)$ for all $\begin{matrix} 0 \le p \le N-1 \\ 0 \le q \le M-1 \end{matrix}$

$\therefore$ DFT of convolution can be reduced to simple multiplication!

Recall: Shift-invariant image transformation = 2D convolution.

$\therefore$ Easy computation/manipulation of shift-invariant transf.

after DFT!!

Proof:

DFT of $g * w$ at (p, q)

$$= \frac{1}{NM} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} g*w(n,m)\, e^{-j2\pi\left(\frac{pn}{N} + \frac{qm}{M}\right)}$$

$$= \frac{1}{NM} \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} g(n-n', m-m')\, w(n', m')\, e^{-j2\pi\left(\frac{pn}{N} + \frac{qm}{M}\right)}$$

Change of variables:

$n \rightarrow n'' = n - n'$

$m \rightarrow m'' = m - m'$

$$= \underbrace{\frac{1}{NM} \sum_{n'=0}^{N-1} \sum_{m'=0}^{M-1} w(n', m')\, e^{-j2\pi\left(\frac{pn'}{N} + \frac{qm'}{M}\right)}}_{\hat{w}(p,q)} \underbrace{\sum_{n''=-n'}^{N-1-n'} \sum_{m''=-m'}^{M-1-m'} g(n'', m'')\, e^{-j2\pi\left(\frac{pn''}{N} + \frac{qm''}{M}\right)}}_{T(p,q)}$$

Note that: g and w are periodically extended.

$\therefore\ g(n-N, m) = g(n, m)$ and $g(n, m-M) = g(n, m)$

$$\therefore\ T \equiv \sum_{m''=-m'}^{M-1-m'} e^{-j2\pi \frac{qm''}{M}} \sum_{n''=-n'}^{-1} g(n'', m'')\, e^{-j2\pi \frac{pn''}{N}} + \sum_{m''=-m'}^{M-1-m'} e^{-j2\pi \frac{qm''}{M}} \sum_{n''=0}^{N-1-n'} g(n'', m'')\, e^{-j2\pi\left(\frac{pn''}{N}\right)}$$

Consider $\sum_{n''=-n'}^{-1} g(n'', m'') e^{-j2\pi\frac{pn''}{N}} \underset{n'''=N+n''}{=} \sum_{n'''=N-n'}^{N-1} \underbrace{g(n'''-N, m'')}_{g(n''', m'')} e^{-j2\pi\left(\frac{pn'''}{N}\right)} \cancel{e^{j2\pi p}}^{\,1}$

$e^{-j2\pi\left(\frac{pn''}{N}\right)}$, $n'' = n'''-N$

We can do similar thing for index m''.

$$\therefore\ T = \sum_{m''=0}^{M-1} \sum_{n''=0}^{N-1} g(n'', m'') e^{-j2\pi\left(\frac{pn''}{N} + \frac{qm''}{M}\right)} = MN\,\hat{g}(p,q)$$

$$\therefore\ \widehat{g * w}\,(p,q) = MN\,\hat{g}(p,q)\,\hat{w}(p,q)$$

Remark: Conversely, if $x(n,m) = g(n,m)\,w(n,m)$

Then, $\hat{x}(k,l) = \sum_{p=0}^{N-1} \sum_{q=0}^{M-1} \hat{g}(p,q)\,\hat{w}(k-p,\, l-q)$ (Convolution of g and w)

2. Average value of image

Average value of $g = \bar{g} = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l) = \underbrace{\frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi(0)}}_{\hat{g}(0,0)}$

3. DFT of a rotated image

Consider a $N \times N$ image g.

Then: $\hat{g}(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l)\, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$

Write k and l in polar coordinates:

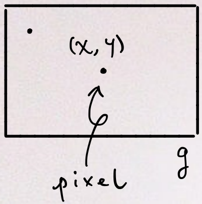
$k \equiv r\cos\theta$; $l \equiv r\sin\theta$

Similarly, write $m \equiv \omega\cos\phi$; $n = \omega\sin\phi$.

Note that: $km + ln = r\omega(\cos\theta\cos\phi + \sin\theta\sin\phi) = r\omega\cos(\theta-\phi)$.

Denote $\mathcal{P}(g) = \{(r,\theta) : (r\cos\theta, r\sin\theta) \text{ is a pixel of } g\}$

(Polar coordinate set of g)



If $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, then $(r, \theta) \in \mathcal{P}(g)$.

Then: $\underbrace{\hat{g}(m, n) = \hat{g}(\omega, \phi)}_{\text{Identify } \hat{g}(m,n) \text{ with } \hat{g}(\omega,\phi)} = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \underbrace{g(r, \theta)}_{\text{Identify } g(k,l) \text{ with } g(r,\theta)} e^{-j2\pi\left(\frac{r\omega\cos(\theta-\phi)}{N}\right)}$

Consider a rotated image $\tilde{g}(r, \theta) = g(r, \theta + \theta_0)$ where θ is defined between $-\theta_0$ to $\pi/2 - \theta_0$.

$\therefore$ image g is rotated clockwisely by θ_0.

DFT of $\tilde{g}$ is:

$$\hat{\tilde{g}}(\omega, \phi) = \frac{1}{N^2} \sum_{(r,\theta)\in\mathcal{P}(\tilde{g})} \underbrace{\tilde{g}(r, \theta)}_{g(r,\, \underbrace{\theta+\theta_0}_{\tilde{\theta}})} e^{-j2\pi\left(\frac{r\omega\cos(\theta-\phi)}{N}\right)} = \frac{1}{N^2} \sum_{(r,\tilde{\theta})\in\mathcal{P}(g)} g(r, \tilde{\theta})\, e^{-j2\pi\left(\frac{r\omega\cos(\tilde{\theta}-\theta_0-\phi)}{N}\right)}$$

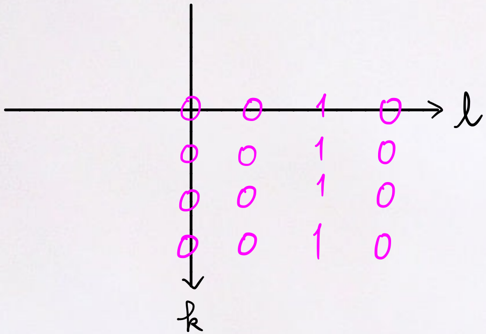
$\therefore \hat{\tilde{g}}(\omega, \phi) = \hat{g}(\omega, \phi + \theta_0)$. (ϕ is also defined between $-\theta_0$ to $\pi/2 - \theta_0$)

DFT of an image rotated by θ_0 = DFT of the original image rotated by θ_0.

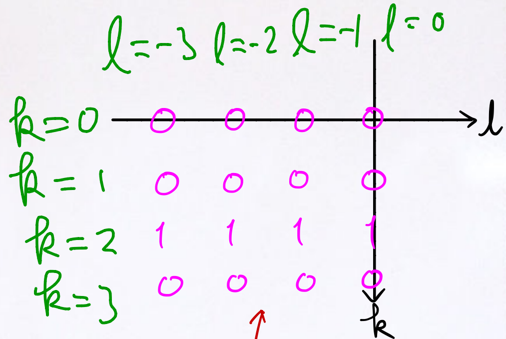
Example: Let $g = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Then: $\hat{g} = $

$$\begin{array}{c c} \begin{array}{cccc} l=0 & l=1 & l=2 & l=3 \end{array} & \\ \begin{pmatrix} \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} k=0 \\ k=1 \\ k=2 \\ k=3 \end{array} \end{array}$$

Note that g in the coordinate system:



Rotated by 90° clockwisely



$\tilde{g}$

Note that indices of $\tilde{g}$ are taken as: $\begin{cases} -3 \leq l \leq 0 \\ 0 \leq k \leq 3 \end{cases}$.

Now, DFT of $\tilde{g} = \hat{\tilde{g}}$ (given by: $\sum_{k=0}^{3}\sum_{l=-3}^{0} \tilde{g}(k,l)\, e^{-j2\pi\left(\frac{km+ln}{4}\right)}$

$$
= \begin{array}{c|c}
\begin{pmatrix} 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & -1/4 \\ 0 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & -1/4 \end{pmatrix} & \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \downarrow k \\
\begin{array}{cccc} -3 & -2 & -1 & 0 \end{array} \xrightarrow{l} &
\end{array}
$$

$\therefore$ $0 \le k \le 3$

$-3 \le l \le 0$

4. DFT of a shifted image

Let $g = (g(k', l'))$ be a $N \times N$ image, where the indices are taken as:

$-k_0 \le k' \le N-1-k_0$ and $-l_0 \le l' \le N-1-l_0$

Let $\tilde{g}$ be shifted image of g defined as:

$\tilde{g}(k,l) = g(k-k_0,\ l-l_0)$ where $0 \le k \le N-1$

$0 \le l \le N-1$

Then: $$\hat{\tilde{g}}(m,n) = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k-k_0,\ l-l_0)\, e^{-j2\pi\left(\frac{km+ln}{N}\right)}$$

$$= \underbrace{\frac{1}{N^2}\sum_{k'=-k_0}^{N-1-k_0}\sum_{l'=-l_0}^{N-1-l_0} g(k',l')\, e^{-j2\pi\left(\frac{k'm+l'n}{N}\right)}}_{\hat{g}(m,n)}\, e^{-j2\pi\left(\frac{k_0 m+l_0 n}{N}\right)}$$

$$\therefore \quad \hat{\hat{g}}(m,n) = \hat{g}(m,n)\, e^{-j2\pi\left(\frac{k_0 m + l_0 n}{N}\right)}$$

__Remark:__ $\hat{g}(m-m_0,\, n-n_0) = \mathrm{DFT}\left(g \times e^{j2\pi\left(\frac{m_0 k + n_0 l}{N}\right)}\right)$ with carefully chosen indices!

Mathematics of JPEG

Consider a $N \times N$ image f. Extend f to a $2M \times 2N$ image $\tilde{f}$, whose indices are taken from $[-M, M-1]$ and $[-N, N-1]$.

Define $f(k,l)$ for $-M \le k \le M-1$ and $-N \le l \le N-1$ such that

$$f(-k-1,-l-1) = f(k,l) \quad \} \text{ Reflection about } (-1/2,-1/2)$$

$$\left.\begin{array}{c} f(-k-1,l) = f(k,l) \\ f(k,l-1) = f(k,l) \end{array}\right\} \text{Reflection about the axis } k=-1/2 \text{ and } l=-1/2$$

Example:

$l = -1/2$

9	8	7	7	8	9	$k=-3$
6	5	4	4	5	6	$k=-2$
3	2	1	1	2	3	$k=-1$
3	2	1	1	2	3	$k=0$
6	5	4	4	5	6	$k=1$
9	8	7	7	8	9	$k=2$
$l=-3$	$l=-2$	$l=-1$	$l=0$	$l=1$	$l=2$	

$k = -1/2$

$f(-2,-2)$
$=$
$f(1,1)$

Reflection about $(-1/2,-1/2)$.

$f(-1,1)$
$=$
$f(0,1)$

Reflection about the axis $k = -1/2$.

Make the extension as a reflection about (0, 0), the axis $k=0$ and the axis $l=0$.

Done by shifting the image by $(1/2, 1/2)$

After shifting

$$
\left.\begin{array}{ccc|ccc|l}
9 & 8 & 7 & 7 & 8 & 9 & \frac{1}{2}+(-3) \\
6 & 5 & 4 & 4 & 5 & 6 & \frac{1}{2}+(-2) \\
3 & 2 & 1 & 1 & 2 & 3 & \frac{1}{2}+(-1) \\
\hline
3 & 2 & 1 & 1 & 2 & 3 & \frac{1}{2}+0 \\
6 & 5 & 4 & 4 & 5 & 6 & \frac{1}{2}+1 \\
9 & 8 & 7 & 7 & 8 & 9 & \frac{1}{2}+2
\end{array}\right\} k
$$

$$
\underbrace{\begin{array}{cccccc}
\frac{1}{2}+-3 & \frac{1}{2}+-2 & \frac{1}{2}+-1 & \frac{1}{2}+0 & \frac{1}{2}+1 & \frac{1}{2}+2
\end{array}}_{l}
$$

Now, we compute the DFT of (shifted) $\tilde{f}$:

$$F(m,n) = \frac{1}{(2M)(2N)} \sum_{k=-M}^{M-1} \sum_{l=-N}^{N-1} f(k,l) e^{-j\frac{2\pi}{2M}m(k+\frac{1}{2})} e^{-j\frac{2\pi}{2N}n(l+\frac{1}{2})}$$

$$= \frac{1}{4MN} \sum_{k=-M}^{M-1} \sum_{l=-N}^{N-1} f(k,l) e^{-j(\frac{\pi}{M}m(k+\frac{1}{2})+\frac{\pi}{N}n(l+\frac{1}{2}))}$$

$$= \frac{1}{4MN}(\underbrace{\sum_{k=-M}^{-1} \sum_{l=-N}^{-1}}_{A_1} + \underbrace{\sum_{k=-M}^{-1} \sum_{l=0}^{N-1}}_{A_2} + \underbrace{\sum_{k=0}^{M-1} \sum_{l=-N}^{-1}}_{A_3} + \underbrace{\sum_{k=0}^{M-1} \sum_{l=0}^{N-1}}_{A_4})$$

$$f(k,l) e^{-j(\frac{\pi}{M}m(k+\frac{1}{2})+\frac{\pi}{N}n(l+\frac{1}{2}))}$$

After some messy simplication, we can get:

$$A_1 + A_2 + A_3 + A_4 = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} f(k,l) \cos\left[\frac{m\pi}{M}\left(k+\frac{1}{2}\right)\right] \cos\left[\frac{n\pi}{N}\left(l+\frac{1}{2}\right)\right]$$

Definition: (Even symmetric discrete cosine transform [EDCT])

Let f be a $M \times N$ image, whose indices are taken as $0 \leq k \leq M-1$ and $0 \leq l \leq N-1$. The **even symmetric discrete cosine transform (EDCT)** of f is given by:

$$\hat{f}_{ec}(m,n) = \frac{1}{MN}\sum_{k=0}^{M-1}\sum_{l=0}^{N-1} f(k,l)\cos\left[\frac{m\pi}{M}\left(k+\frac{1}{2}\right)\right]\cos\left[\frac{n\pi}{N}\left(l+\frac{1}{2}\right)\right]$$

with $0 \leq m \leq M-1, 0 \leq n \leq N-1$

Remark:
- Smart idea to get a decomposition consisting only of cosine function (by reflection and shifting!)
- Can be formulated in matrix form
- Again, it is a separable image transformation.

- The inverse of EDCT can be explicitly computed. More specifically, the **inverse EDCT** is defined as:

$$f(k,l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} C(m)C(n)\hat{f}_{ec}(m,n) \cos \frac{\pi m(2k+1)}{2M} \cos \frac{\pi n(2l+1)}{2N} \qquad (**)$$

where $C(0) = 1, C(m) = C(n) = 2$ for $m, n \neq 0$

Also involving cosine functions only!

- Formula (**) can be expressed as matrix multiplication:

$$f = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \hat{f}_{ec}(m,n) \vec{T}_m \vec{T'}_n^T$$

elementary images under EDCT!

where: $\vec{T}_m = \begin{pmatrix} T_m(0) \\ T_m(1) \\ \vdots \\ T_m(M-1) \end{pmatrix}, \vec{T'}_n = \begin{pmatrix} T'_n(0) \\ T'_n(1) \\ \vdots \\ T'_n(N-1) \end{pmatrix}$ with $T_m(k) = C(m) \cos \frac{\pi m(2k+1)}{2M}$

This is what JPEG does!!

and $T'_n(k) = C(n) \cos \frac{\pi n(2k+1)}{2N}$.

Something similar can be developed:

Definition: (Odd symmetric discrete cosine transform [ODCT])

Let f be a $M \times N$ image, whose indices are taken as $0 \leq k \leq M-1$ and $0 \leq l \leq N-1$. The **odd symmetric discrete cosine transform (ODCT)** of f is given by:

$$\hat{f}_{oc}(m,n) = \frac{1}{(2M-1)(2N-1)} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} C(k)C(l)f(k,l) \cos\frac{2\pi mk}{2M-1} \cos\frac{2\pi nl}{2N-1}$$

where $C(0) = 1$ and $C(k) = C(l) = 2$ for $k, l \neq 0$, $0 \leq m \leq M-1, 0 \leq n \leq N-1$.

The **inverse ODCT** is given by:

$$f(k,l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} C(m)C(n)\hat{f}_{oc}(m,n) \cos\frac{2\pi mk}{2M-1} \cos\frac{2\pi nl}{2N-1}$$

where $C(0) = 1, C(m) = C(n) = 2$ if $m, n \neq 0$